The Pre T ½ Spaces (The New Further Results)

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Abstract:

We give the equivalence between pre T ½ space and some types of mappings. The aim of this paper is to discuss and investigate some characterizations for pre T ½ spaces. Also, the implication of this notion among themselves and with the well-known axioms such as : pre-T_0, pre-T_1 and pre-T_2 are introduced. Furthermore we introduce and study the definition of preT_D and pre-symmetric spaces and some of their properties are discussed.

1. Introduction And Preliminaries

In 1982, Mashhour et al. [6] introduced the notion of a preopen set so, many mathematicians turned their attention to the generalizations of various concepts of topology by considering preopen sets instead of open sets. In this way, Maki, Umehara and Noiri [5] define the concept of pre-generalized closed sets of a topological space taking help of the preopen sets. In the present paper, we continue to give some characterizations for pre T ½ spaces.

Also, we introduce the equivalence between a pre T ½ space and some types of mappings.

Further, we discuss and investigate the definition of pre-symmetric and pre-T_D-spaces and some of their properties are introduced.

Throughout this paper (X,τ), (Y,σ) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For subset B of (X,τ), cl(B) and int(B) denotes the closure of B with respect to τ and the interior of B with respect to τ respectively. (X,τ), will be replaced by X if there is no chance of confusion.

Now, we recall the following definitions which we shall require later.

Definition 1.1.

A subset B of a space X is called :

1) a preopen set [6] if $B \subseteq int(cl(B))$ and a preclosed set if $cl(int(B)) \subset B$.
2) an α-open set[9] if $B \subseteq int\ (cl(int(B)))$ and an α-closed set [8] if $cl)int(cl(B))) \subset B$

The intersection of all preclosed (resp.α-closed) sets containing a subset B of X is called preclosure (resp. α-closure) of B and is denoted by pcl(B) (resp.αclosure) of B and id denoted by pcl(B). (resp.αcl(B)). The preinterior (resp. α-interior) of a subset B of N is the largest preopen (resp.α-open_ set contained in B and is denoted by pint(B) (resp.α-omt(B))). The Family of all preopen (resp.α-open, preclosed, α-closed) subset of (X,τ) will be denoted by $PO(X,\tau)$ (resp.$\alpha O(X,\tau)$, $PC(X,\tau)$, $\alpha C(X,\tau)$).

Definition 1.2.

A subset B of a space X is said to be :

1) a generalized closed set (briefly g-closed) [2] if $cl(B) \subseteq U$ whenever $B \subset U$ and U is open in (X,τ),

2) a pre-generalized closed set (briefly pg-closed) [5] if $pcl(B) \subset U$ whenever $B \subset U$ and $\in PO(X,\tau)$.

Definition 1.3.

Recall that a mapping $f : (X,\tau) \longrightarrow (Y, \sigma)$ is called :

1) preirresolute [11] if . $f^{-1}(V) \in PO(X,\tau)$, for every $V \in PO(X,\tau)$, for every $V \in PO(X,\sigma)$,

2) preclosed [3] if for each closed set U of (X,τ) *f*(U) is preclosed in (X,σ).

3) M-preopen [7] if for each preopen set Uof (X,τ), *f*(U) is preopen in (X,σ).

Definition 1.4.

A space (X,τ) is said to be :

1) a pre-$T_{½}$-space [5] if every pg-closed set is preclosed.

2) a pre-T_o (resp.pre0T_1) space [4] if, for $x,y \in X$ such that $x \neq y$, there exists a preopen set containing x but not y or (resp.and) a preopen set containing y but not x.

3) a pre-T_2 –space [4] if, for x, $y \in X$ such that $x \neq y$ there exist preopen sets U_1 and U_2 such $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \varnothing$.

According (2) , (3) from Definition 1.4. the following implications hold :

$$\begin{tikzcd}
T_2 \arrow[r] \arrow[d] & \text{pre-}T_2 \\
T_1 \arrow[r] \arrow[d] & \text{pre-}T_1 \\
T_0 \arrow[r] & \text{pre-}T_0
\end{tikzcd}$$

2. The Bahaviour of PRE – $T_{½}$ Spaces

Theorem 2.1

A set B of a space X is pg-closed if and only if pcl(B)\B does not contain any non-empty preclosed set.

Proof.

Necessity, Obvious from [1, Proposition3.1]

Sufficiency

Let $B \subset U$, where U is preopen in (X,τ).

If pcl(B) is not contained in U, then pcl(B) $\cap$ X/U $\neq$ is a non-empty preclosed set, then we obtain a contracation and therefore B is pg-closed.

Lemma 2.1

If B is a pg-closed set of a space X, then the following are equivalent

1) B is preclosed

2) Pcl(B)\B is preclosed

Proof

(1)$\longrightarrow$ (2) : If B is a pg-closed set which is also preclosed, then by Theorem 2.1.

Pcl(B)\B = $\varnothing$ which is preclosed.

(2)$\longrightarrow$ (1) : Let pcl(B)\B be a preclosed set and B be pg-closed. Then by Theorem 2.1. pcl(B)\B does not contain any non-empty preclosed subset. Since pcl(B)\B is preclosed and pcl(B)\B is preclosed and pcl(B)\B = $\varnothing$, this shows that B is preclosed.

In 1996, Mari, Umehara and Noiri [5] showed that the following theorem.

Theorem 2.2:for space (X,τ), the following are equivalent

1) (X,τ) is a pre- $T_{½}$ -space

2) Every subset of X is the intersection of all preopen sets and all preclosed sets containing it.

Theorem 2.3.

If X is a space then the following statements are equivalent

(1)(X,τ) is a pre- $T_{½}$ -space

(2) Esubset of X is the intersection of all preopen sets and all preclosed sets containing it.

Proof

(1) $\longrightarrow$(2). If X is pre-$T_{½}$ -space with $B\subset X$, then $B=\cap\{X\backslash\{x\}:x\notin B\}$ is the intersection of preopen and preclosed sets containing it.

(2) $\longrightarrow$(1) For each $x\notin X$, then $X\backslash\{x\}$ is the intersection of all preopen and

Preclosed sets containing it, Hence $X\backslash\{x\}$ is either preopen or preclosed.

Therefore by Theorem 2.2, (X,τ) is pre-$T_{½}$

Lemma 2.2.

For a space (X,τ), the following are equivalent :

1) Every subset of X is pg-closed

2) PC (X,τ)= PC(X,τ).

Proof

(1) $\longrightarrow$ (2) : Let $U \in PO(X,\tau)$, Then by hypothesis, U is pg-closed which implies that $pcl(U) \subset U$, therefore $U \in PO(X,\tau)$, Also let $V \in PC(X,\tau)$, hence by hypothesis X/V is pg-closed and then $X \setminus V \in PC(X,\tau)$ thus $V \in PO(X,\tau)$ according above we have PO $U \in (X,\tau)$ = PC $U(X,\tau)$.

(2) $\longrightarrow$ (1) : If B is a subset of a space X such that $B \subset U$ where $U \in PO(X,\tau)$,, then $U \in PC(X,\tau)$, and therefore $pcl(B) \subset U$ which shows that B is pg-closed.

Proposition 2.1

The property of being a pre-$T_{½}$-space is hearedity.

Proof

If Y is a subspace of a pre-$T_{½}$ -space X and $y \in Y \subset X$, then {y} is either preopen or preclosed (X,τ)

(by Theorem 2.2) Therefore {y} is either preopen or preclosed in Y. Hence Y is a pre $T_{½}$ -space.

Theorem 2.4

If (X,τ),, is a topological space. Then the following statements are hold

1) Every pre-T_1-space is pre – $T_{½}$

2) Every pre$T_{½}$ space is pre-T_o

Proof

1) Since every pre-T1 space is a topological space then by [5, Theorem 2.27] it follows that X is pre-$T_{½}$

2) Since every pre – $T_{½}$ space is a topological space then by [4.Theorem3] it follows that X is pre - T_o

According the above theorem, the following implication hold;

Pre-T_1-space $\longrightarrow$ pre-$T_{½}$- space $\longrightarrow$ preT_0 –space

but the converse is not true as is shown in the following examples.

Example 2.1

If X = {a,b,c} with $\tau=\{\varnothing, \{a\},X\}$ then (X,τ) is a pre – $T_{½}$ -space but it is not pre-T_0 Since {a} is not preclosed.

Example 2.2.

Let X =Pa,b} with $\tau = \{\varnothing, \{a\},X\}$, Then (X, τ) is a pre-T_0-space but it is not pre-$T_{½}$

3. **Characterization of PRE-$T_{½}$ Spaces**

In this section, we give the definition of approximately preirresolute and approximately preclosed mappings. Also, we introduce some characterizations of pre $T_{½}$ -spaces on these mappings which are mentioned above.

Definition 3.1

A mapping f:(X, τ) $\longrightarrow$ (Y, σ) is called.

1) Approximately preirresolute (briefly, ap-pre-irresolute). If pcl(V) $\subseteq$ f^1(H) whenever H is a preopen subset of (Y,σ), V is a pg-closed subset of (X, τ) and V$\subseteq f^{-1}$H.

2) Approximately preclosed (briefly, ap-preclosed) if f(A) $\subseteq$ pint(H), whenever H is a pg-open subset of (Y,σ),A is a preclosed subset of:(X, τ)and f(A) $\subseteq$H

Theorem 3.1

If (X, τ) is apre-T $_{½}$ -space and f :(X, τ) -- (Y, σ) ,surjective preirresolute and M-preopen.

Proof :

Let A $\subseteq$ Y be a pg-closed set. Then we will prove that f^{-1}(A) is pg-closed in (X, τ) If f^{-1}(A) $\subseteq$ H, where H is preopen in (X, τ),

If f^{-1}(A)$\subset$ H ,where H is preopen in (X, τ)

Now,

f(pcl(f^{-1}(A))$\cap$ X\H) $\subseteq f$(pcl(f^{-1}(A)))$\cap f$(X\H)

$\subseteq f$(pcl(f^{-1}(A)))$\cap$X\A

$\subseteq f$(pcl($f\,f^{-1}$(A)))$\cap$X\A

$\subseteq$ pcl(A)$\cap$X\A

By ([1] **Proposition 3.1**)

F(pcl(f^{-1} (A)) $\cap$ X\A= $\varnothing$, hence

Pcl(f^{-1}(A))$\cap$X\H = $\varnothing$, Then

Pcl(f^{-1} (A)) $\subseteq$ H, Therefore,

f^{-1} (A) is pg-closed in (X,τ), where (X,τ) iis pre-T $_{½}$

then f^{-1}(A) is preclosed in (X,τ).

Hence A=f(f^{-1}(A) is preclosed in (Y, σ) is pre-T $_{½}$

(Y, σ) is pre-T $_{½}$ -space.

Theorem 3.2.

For a topological space (Y, τ), the following are equivalent :

1) (X, τ) is a pre-T $_{½}$ space

2) There exists a mapping f: X-Y such that f is ap-preirresolute

Proof

(1) => (2): Let V be a pg-closed subset of (X, τ) and assume that there exists a mapping f: X-Y such that V $\subseteq f^{-1}$(G), where H is a preopen set of (Y,σ) Since (X, τ) is a pre-T $_{½}$ -space, then V is preclosed, hence V=pcl(V), Therefore pcl(V) $\subseteq f^{-1}$(H) Then f is ap-preirresolute.

(2) => (1): Let A be a pg-closed subset of (X, τ) and let Y be the set X with the topology σ ={$\varnothing$, A, Y} Also, let f: (X, τ) $\longrightarrow$ (Y, σ) be the identify

map. Since A is pg-closed in (X, τ), preopen in (X, τ) and A ⊆ f-1(A). Since f is ap-preirresolute then pcl(A) ⊆ f-1(A) =)A). therefore A is preclosed in (X, τ) and hence (X, τ) is a pre-T $_{½}$ space.

Theorem 3.3.
For a space (Y,σ), the following are equivalent :
1- (Y, σ) is pre-T $_{½}$ -space.
2- There exists a function f: X-Y such that f is ap-preclosed.
Proof
Analogous to Theorem 3.2, making the obvious changes.
4. PRE –Symmetric spaces
In this section we, introduce the definition of a symmetric and a pre-T_D spaces. Also, some of their properties are discussed.
Definition 4.1
A topological space (X, τ) is called a pre-symmetric space if, for X and Y in X, X ∈ pcl({x}).
Example 4.1
Let X = {a, b, c, d} with τ = {∅, {a,b}, {c,d},X} Then (X, τ) is a pre-symmetric space.
Example 4.2
A Sierpinski space is not pre-sysmmetric, since 1 ∈ pcl({10}) does not imply that 0 ∈ pcl({1})
Theorem 4.1
If (X, τ) is a topological space, then the following are equivalent :
1) (X, τ) is a pre-symmetric space.

2) Every singleton is pg-closed, for each x∈X

Proof.
(1) => (2) : Assume that {x}⊂ U ∈ PO(X, τ), but pcl({x}) ⊄ U. Then pcl({x}) ∩ X\U ≠∅. Now we take y ∈ pcl({x}) ∩ X\U, then by hypothesis X ∈ U which is a contradiction. Therefore {x} is pg-closed, for each x∉X.

(2) => (1), Assume that x ∈ pcl ({y}), but y ∉ pcl({x}). Then {y} ⊂X\pcl{x}) and hence pcl({y}) ⊂ X\pcl({x}). Therefore x ∈X\pcl({x}).

Corollary 4.1
If (X, τ) is a pre-T_1-space, then (X, τ) is pre-symmetric.
Proof
Since (X, τ) is a pre-T1-space, then every singleton is preclosed and therefore is pg-closed. Then by Theorem 4.1 (X, τ) is pre-symmetric.

Lemma 4.1.

For a space (X, τ), the following statements are equivalent

1) (X, τ) is pre-symmetric and pre-T_o

2) (X, τ) pre-T_1

Proof

The proof it follows from Theorem 3.3[10]

Theorem 4.2

If (X, τ) is a pre-symmetric space, then the following statements are equivalent

1) (X, τ) is a pre-T_o – space.
2) (X, τ) is a pre-$T_{½}$
3) (X, τ) is a pre-T_1 space

Proof

(1) $\Leftrightarrow$ (3) obvious from Lemma 4.1

(2) (3)=> (2) => (1): Directly from Theorem 2.1.

Definition 4.2

A topological space (X, τ) is called a pre –T_D-space if every singleton in either open or nowhere dense (or equivalently the derived set $cl(\{x\})\backslash\{x\}$ is preclosed, for each point $x \in X$).

Example 4.3.

If $X = \{p,q\}$ with $\tau = \{\emptyset, \{q\}, X\}$, then (X, τ) is a pre-T_D-space.

Example 4.4.

A space (X, τ) in Example 4.1is not pre-T_D since a subset $\{a\}$ is neither open nor nowhere dense.

Remark 4.1

According the above examples, we note that a pre-symmetric space and a pre-T_D-space are independent.

Theorem 4.3.

For a topological space (X, τ) the following are equivalent :

1) (X, τ) is a preT_D-space.

2) (X, τ) is a pre$T_{1/2}$ –space

Proof

It follows from [5,Theorem 2.27]

Corollary 4.2.

If (X, τ) is a pre-symmetric space, then the following are equivalent :

1) (X, τ) is a preT_D-space

2) (X, τ) is a preT_o-space

3) (X, τ) is a preT_1-space

Conclusion

The initiation of introduced of the notion of a preopen set by Mashhour et al. [6] in 1982. Many mathematicians turned their attention to the generalizations of various concepts of topology by considering preopen sets instead of open sets.

By this way, Maki, Umehara and Noiri [5] introduced the concept of pre-generalized closed sets of a topological space taking help of the preopen sets. In this paper, we continue to give some properties for pre-$T_{½}$-spaces which introduced by Maki et al. [5]. Also, we introduced the definition of a pre-symmetric and a pre-T_D-spaces.

Further some characterizations of these spaces are investigated. Furthermore, we give the definition of ap-preirresolute and ap-preclosed mappings and we discuss the relation between these maps and pre-$T_{½}$-spaces.

Finally, we prove that a pre-symmetric space and pre- $T_D$space are independent.

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النتائج المستقبلية الجديدة للفضاء التبولوجيّ Pre T ½

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الملخص:

نحن نعطي التكافؤ بين الفضاء قبل **T ½** وبعض أنواع التطبيقات. وهذا البحث هو مناقشة والتحقيق في بعض الأوصاف لمرحلة مسافات ما قبل **T ½** و الآثار المترتبة على هذه الفكرة فيما بينها أو مع البديهيات المعروفة كمثل : يتم إدخال ما قبل T0 ، قبل T1 و T2 قبل. وعلاوة على ذلك نحن نقدم دراسة تعريف preTD والمساحات قبل التماثل وبعض ممتلكاتهم و مناقشتها.